DIFFUSION–ABSORPTION EQUATION WITH GENERAL DATA

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1. Introduction. The aim of this paper is to present some results to the Cauchy problem

$$u_t = \Delta\varphi(u) - g(u), \tag{1}$$

posed in the domain

$$S := R^N \times (0, +\infty),$$

subject to the initial data

$$u(x,0) = u_0(x), \qquad \text{for all } x \in R^N. \tag{2}$$

The initial data u_0 is a nonnegative and locally integrable function in R^N. No assumption has to be made on the behavior of u_0 as $|x|$ tends to infinity. The function φ and g should satisfy the following conditions.

a) $\varphi(0) \geq 0, \varphi'(u) > 0 \forall u > 0, \varphi \in C[0,+\infty) \cap C^{2+\alpha}(0,+\infty), \quad 0 < \alpha < 1,$

b) $g(0) = 0, g(u) > 0 \forall u > 0, g \in C[0,+\infty) \cap C^1(0,+\infty),$

c) $\varphi(+\infty) = +\infty$, $g \circ \varphi^{-1}$ is convex on R_+, and

$$F(u) = \int_{u^*}^{\infty} \left[\int_0^y g \circ \varphi^{-1}(s) ds\right]^{\frac{1}{2}} dy < \infty, \quad u^* > \varphi(0). \tag{3}$$

Here φ^{-1} is the inverse function for φ. It is not difficult to check that all conditions are true for example for the following functions:

1) $\varphi(u) = \exp(\alpha u), \quad g(u) = \exp(\beta u) - 1, \quad \beta > \alpha > 0;$

2) $\varphi(u) = u^m, \quad g(u) = u^p, \quad p > m > 0;$

3) $\varphi(u) = \ln^\alpha(u+1), \quad g(u) = \ln^\beta(u+1), \quad \beta > \alpha > 0;$

4) $\varphi(u) = u^m, \quad g(u) = u^m \ln^\alpha(u+1), \quad m > 0, \alpha > 2.$

Equation (1) is encountered, for example, in a process of thermal propagation accompanied by absorption in the case where the thermal conductivity and the absorption coefficient are assumed to be dependent on the temperature represented by u. It is to be noticed that (1) includes some cases of "slow " as well as " fast" diffusion equations.

As it is well known problem (1)–(2) cannot have a classical solutions even when the initial data is smooth.

Definition 1. A nonnegative function u is said to be a weak solution of (1)–(2) if $u \in C([0,+\infty); L^1_{loc}(R^N))$ and satisfies (1) in $D'(S)$ and (2).

DEFINITION 2. We call u a strong solution to (1)–(2) if u is a weak solution and a continuous function for $t > 0$.

A large number of papers have been devoted to equations of the type (1) (see for example [9] and the references therein). The existence and uniqueness of solutions under such general initial data is not the common case in parabolic problems. We point out some close papers. The Cauchy problem for the following equation

$$u_t = \Delta u^m - cu^p, \qquad (x,t) \in S, \tag{4}$$

where $m > 1, p > 1$ and c is a positive constant, were considered in [6], [10]-[13]. In particular, in [6] and [12] for $p > m$ it was shown that the Cauchy problem has a unique solution without any growth condition at infinity on the initial data. The case of equation (4) with variables coefficient $c(x,t)$ was considered in [7]. The results in the same direction was obtained by Brezis in the semilinear case $m = 1, p > 1$ with constant coefficient, cf. [2]. A different example is provided by the fast diffusion equation $u_t = \Delta u^m, 0 < m < 1$, studied by Herrero and Pierre [8]. The same questions for the Cauchy problem for generalized complex Ginzburg–Landau equation were considered in [4], [5]. The close problems for elliptic equations were investigated in papers [2], [3].

The main purpose of the present paper is to get the existence of a weak solution and the existence and the uniqueness of a strong solution to (1)–(2), without any growth restriction at an infinity on the initial data. Under some additional conditions we investigate the boundness for any $t \geq \tau > 0$ as well as vanishing for finite time for a strong solution. The Dirichlet problem for equation (1) with infinite boundary values is considered too.

2. Main Results. First we consider the existence of a weak solution to problem (1)–(2) where the initial data u_0 is a nonnegative function in $L^1_{loc}(R^N)$. Let $B_R := \{x \in R^N : |x| < R\}$. We shall write $\int_{B_R} f$ for $\int_{B_R} f(x)dx$. We use a regularization process and some apriori estimate to establish the existence of a weak solution to (1)–(2).

THEOREM 1. *For every nonnegative function $u_0 \in L^1_{loc}(R^N)$, problem (1)–(2) admits a weak solution u satisfying*

$$\int_{B_R} u(t) + \frac{1}{2}\int_0^t \int_{B_R} g(u) \leq C\left[\int_{B_R} u_0 + tR^{N-2}\right], \tag{5}$$

where $t > 0, R > 1$ and $C = C(N)$.

To prove the existence of a strong solution to (1),(2) we need the following additional hypothesis

$$\lim_{s\downarrow 0} \varphi'(s) = A \leq +\infty, \tag{6}$$

and

$$u_0 \in L^\infty_{loc}(R^N) \tag{7}$$

or for some positive constant a

$$\int_a^\infty \frac{d\tau}{g(\tau)} < \infty. \tag{8}$$

THEOREM 2. *Suppose in addition to assumptions of Introduction that* (6) *and* (7) *or* (6) *and* (8) *hold. Then problem* (1), (2) *admits a strong solution* u. *Moreover* $u(x,.) \in L^\infty[0,\infty)$ *for any* $x \in R^N$ *or* $u \in L^\infty(R^N \times [\tau,\infty))$ *for any* $\tau > 0$ *respectively. And finally if* u_0 *is continuous function then* $u \in C(\overline{S})$.

It may be of interest to note that the results of Theorem 2 are valid without the assumption on convexity of $g(\varphi^{-1})$. The bounded for $t \geq \tau > 0$ of supersolution to (1)–(2) with additional condition on the nonlinearities of the equation and with continuous initial data in one dimensional case was constructed in [1].
Using the regularization process of the constructed a strong solution to problem (1)–(2) we establish the uniqueness of the strong solution of this problem.

THEOREM 3. *Problem* (1)–(2) *has no other strong solution beside that whose existence was established in Theorem* 2.

Now we shall consider the behaviour of the strong solution as $t \to \infty$. Theorems 2 and 3 state that problem (1), (2) possesses a unique strong solution u. For our exact formulation we need in a new condition on the nonlinearities of the equation:

$$\lim_{s\to\infty}\left(\sqrt{\varphi(s)}\right)' \sqrt{g(s)} = \infty. \tag{9}$$

THEOREM 4. *Let in addition to assumptions of Introduction that* (6), (7), (9) *or* (6), (8) *and*

$$\int_0^\infty \frac{ds}{g(s)} = +\infty. \tag{10}$$

are fulfilled. Then the unique strong solution u *to* (1), (2) *is bounded for any* $t \geq \tau > 0$ *and* $u(x,t)$ *goes to 0 as* t *tends to infinity.*

Under some additional hypothesis we deduce a sufficient condition such that the strong solution vanishes identically after a finite time.

THEOREM 5. *Assume that conditions of Theorem 4 except of* (10) *are valid. Assume in addition the following*

$$\int_0^b \frac{ds}{g(s)} < +\infty,$$

for some number $b > 0$, *then*

$$u(.,t) \equiv 0, \qquad \textit{for all } t \geq T_0.$$

It easy to verify that all conditions of Theorem 5 are valid for example for the following equations:

$$u_t = \Delta \exp(\alpha u) - \exp(\beta u) + 1 - g(u),$$
$$u_t = \Delta u^m - u^p - g(u),$$

where $m > 0, p > \max(m,1), \beta > \alpha$ and $g(u) = u^q, 0 < q < 1$ or $g(u) = u \ln^\alpha(u+1)$for $u > 0, g(0) = 0, -1 < \alpha < 0$. Hence the strong solution of the Cauchy problem for these equations with any nonnegative locally integrable initial data vanishes identically after some finite time.

We study also the existence of a strong solution to initial and infinite boundary problem for equation (1). More precisely, we consider the problem

$$\begin{cases} u_t = \Delta\varphi(u) - g(u), & \text{in } Q := \Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ \lim_{y\in\Omega, y\to x\in\partial\Omega} u(x,t) = +\infty, & \text{on } \Gamma := \partial\Omega \times (0,+\infty), \end{cases} \tag{11}$$

where Ω is domain in R^N with smooth boundary. We suppose that the initial data, u_0, is nonnegative and continuous function. The problem of such kind has been investigated in [12] for equation (4) with $p > m$.

THEOREM 6. *Suppose that* $u_0 \geq 0$ *and* $u_0 \in C(\Omega)$. *Then problem* (11) *has a strong solution* $u \in C(\Omega \times [0,+\infty))$.

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